The Pennsylvania State University

Department of Economics

Econ 390, Section 101, Summer 2007

Final examination - SOLUTION

You have **1 hour and 50 minutes** to complete the exam. You are not allowed to use any textbooks, notes and etc., except the list of the formulas provided. You may use calculators, but not PDAs or laptops or cell phones. No communication with other students is allowed. You can earn up to **115** point on this exam. However, **15** points is a bonus and is not required to get an A for the course. Each problem has its value in points.

Please show your work step by step! Use the space provided to answer questions.

Problem 1 (10 points)

A local car dealership has two models of cars available: Toyota Camry and Nissan Primera. This Monday while you are sitting here and writing this exam, the local dealership has 5 Toyotas and 4 Nissans for sale.

Somewhere close to 9 am, Jeff enters the dealership with an intent to purchase four vehichles. He says that he does not care on particular producer and is going to choose 4 cars RANDOMLY.

a. Find the probability that he is going to end up with exactly 2 Toyotas and 2 Nissans.

b. Find the probability that all cars he purchases are Toyotas.

c. Find the probability that he purchases not more than 2 Nissans.

a. The total number of ways to choose 4 cars out of 9 available is:

$$C_9^4 = \frac{9!}{4!(5-4)!} = 126$$

The total number of ways to choose 2 Toyotas out of 5 available is:

$$C_5^2 = \frac{5!}{2!(5-2)!} = 10$$

The total number of ways to choose 2 Nissans out of 4 available is:

$$C_4^2 = \frac{4!}{2!(4-2)!} = 6$$

$$P(2\ Toyotas \cap 2\ Nissans) = \frac{C_5^2 * C_4^2}{C_9^4} = \frac{10 * 6}{126} \approx 0.4762$$

b.

$$P(All\ Toyotas) = \frac{C_5^4 C_4^0}{C_9^4} = \frac{5}{126} \approx 0.0397$$

c.

$$P(Not\ more\ than\ 2\ Nissans) = P(All\ Toyotas) + P(1\ Nissan) + P(2\ Nissans) =$$

$$= \frac{C_5^4 C_4^0}{C_9^4} + \frac{C_5^3 C_4^1}{C_9^4} + \frac{C_5^2 C_4^2}{C_9^4} = 0.0397 + 0.3175 + 0.4762 \approx 0.8334$$

Problem 2 (20 points)

Ivan is grading this exam. He finds that the probability that a student solves the first problem correctly is 0.8.

The probability that a student solves the second problem correctly is 0.9, but it is conditional on the event that he solves the first one correctly. Probability that the second problem is solved with a mistake conditional on the event that the first one is incorrect is 0.70.

a. Find the probability that a person who solves the second problem correctly had solved the first problem with no mistakes.

b. Find the probability that a student who solves the second problem incorrect has solved the first problem with some mistake.

(Hint: in part a and part b you have to find CONDITIONAL probabilities. Think how you can use Bayes theorem to solve the problem)

First we have to write down the informationm given:

P(First Correct)=0.8

P(Second Correct | First correct)=0.9

P(Second Incorrect | First Incorrect)=0.7

We are asked to find in the part a P(First Correct | Second Correct) and in the part b P(First Incorrect | Second Incorrect). Do answer these two question we can use Baye's Theorem:

$$P(First\ Correct \mid Second\ Correct) = \frac{P(Second\ Correct \mid First\ correct) * P(First\ Correct)}{P(Second\ Correct)}$$

$$P(First\ Incorrect \mid Second\ Incorrect) = \frac{P(Second\ Incorrect \mid First\ Incorrect) * P(First\ Incorrect)}{P(Second\ Incorrect)}$$

P(Second Correct ∩ First correct)=P(Second Correct | First correct)*P(First Correct)=0.9*0.8=0.72

P(Second Incorrect ∩ First incorrect)=P(Second Incorrect | First Incorrect)*P(First incorrect)=0.7*(1-0.8)=0.14

Using this information we can summirze problem infromation in the following table:

	Second is correct	Second in incorrect	
First is correct	0.72	0.08	0.80
First is incorrect	0.06	0.14	0.20
	0.78	0.22	1.00

a.

$$P(First\ Correct|Second\ Correct) = \frac{P(Second\ Correct \mid First\ correct) * P(First\ Correct)}{P(Second\ Correct)} = \frac{0.9 * 0.80}{0.78} \approx 0.92$$

b.

$$P(First\ Incorrect \mid Second\ Incorrect) = \frac{P(Second\ Incorrect \mid First\ Incorrect) * P(First\ Incorrect)}{P(Second\ Incorrect)} = \frac{0.7 * 0.20}{0.22} \approx 0.64$$

Problem 3(20 points)

University asks all graduate students to take 2 qualification exams: one in microeconomics and one in macroeconomics. For several years results could be summarized by the following joint probability distribution:

		Grade for Microeconomics (Y)					
		70	80	90	$P(X)$	$x * P(x)$	$(x-\mu_x)^2 p(x)$
Grade	60	0.12	0.09	0.07	0.28	16.80	26.3452
for	70	0.07	0.21	0.19	0.47	32.90	0.0423
Macroeconomics (X)	80	0.10	0.02	0.13	0.25	20.00	26.5225
	$P(Y)$	0.29	0.32	0.39	1.00	69.70	52.9100
	$y * P(y)$	19.60	25.60	35.10	80.30		
	$(y-\mu_y)^2 p(y)$	30.7661	0.0288	36.6951	67.4900		

(For instance, probability that $X = 70$ and $Y = 80$ is 0.05 or $P(X = 70 \cap Y = 80) = 0.21$)

a) Write down the marginal probability function $-$ $P(X)$ for X and the marginal probability function $-$ $P(Y)$ for Y (**Hint:** You can use free cells to answer this and the following questions)

See the column 6 for $P(x)$ and the row 6 of the table for $P(y)$

b) Find the mean of X and the mean of Y (Hint: Use your marginal probability functions)

$$E(X) = \sum_{x \in X} (x * P(x)) = 69.7000$$

$$E(Y) = \sum_{y \in Y} (y * P(y)) = 80.30$$

c) Find standard deviation of X and standard deviation of Y , i.e. σ_X and σ_Y

$$\sigma_X = \sqrt{\sum_{x \in X} (x-\mu_X)^2 P(x)} \approx \sqrt{52.9100} \approx 7.2739$$

$$\sigma_Y = \sqrt{\sum_{y \in Y} (y-\mu_Y)^2 P(y)} \approx \sqrt{67.4900} \approx 8.2152$$

e) Find the covariance between X and Y

$$Cov(X,Y) = E(XY) - \mu_X * \mu_Y$$

$$E(XY) = 60 * 70 * 0.12 + 60 * 80 * 0.09 + 60 * 90 * 0.07 +$$

$$+70*70*0.07+70*80*0.21+70*90*0.19+$$

$$+80*70*0.10+80*80*0.02+80*90*0.13=5654$$

$$Cov(X,Y)=E(XY)-\mu_X*\mu_Y=5654-69.7*80.3=57.09$$

f) Find the correlation between X and Y. (I have calculated σ_Y for you: $\sigma_Y=0.3585$)

$$Corr(X,Y)=\frac{Cov(X,Y)}{\sigma_X*\sigma_Y}=\frac{57.09}{7.2739*8.2152}\approx 0.9554$$

d) Find the conditional probability function of X for $Y=70$.

$$P(X=x\mid Y=70)=\frac{P(Y=70\ \cap\ X=x)}{P(Y=70)}$$

$$P(X=60\mid Y=70)=\frac{P(Y=70\ \cap\ X=60)}{P(Y=70)}=\frac{0.12}{0.29}\approx 0.4138$$

$$P(X=70\mid Y=70)=\frac{P(Y=70\ \cap\ X=70)}{P(Y=70)}=\frac{0.07}{0.29}\approx 0.2414$$

$$P(X=80\mid Y=70)=\frac{P(Y=70\ \cap\ X=80)}{P(Y=70)}=\frac{0.10}{0.29}\approx 0.3448$$

f) Use your answer for part d to calculate conditional expectation of X when $Y=70$.

(**Hint**: You have to find $E(X|Y=70)$)

$$E(X|Y=y)=\sum_{x\in X}(x*P(X=x\mid Y=y))=\sum_{x\in X}\left(x*\frac{P(X=x\ \cap\ Y=y)}{P(Y=y)}\right)$$

$$E(X|Y=70)=0.4138*60+0.2414*70+0.3448*80=69.31$$

Problem 4 (15 points)

"Precision Camera Service" is a firm that specializes in digital SLR camera repairs. The firm recorded the number of received cameras on a daily basis for 9 days.

Day	1	2	3	4	5	6	7	8	9
# of cameras received	5	7	11	4	1	3	2	5	6

Assume that number of cameras arriving at a given day has a normal distribution.

(**Hint:** Note, that this assumption does **not** mean that you have to use normal distribution to do the testing, since you do not know population variance)

1. Find 90% confidence interval for the average number of cameras per day that are going to arrive during next year (i.e. for the population mean μ).

2. Test at 5% level H_0: $\mu \geq 5$ against H_1: $\mu < 5$

$$\bar{X} = \frac{5+7+11+4+1+3+2+5+6}{9} \approx 4.8889$$

$$S^2 = \frac{1}{n-1}\sum_{x \in X}(x-\bar{X})^2$$

Observation	5	7	11	4	1	3	2	5	6	$\frac{1}{8}\sum(x-\bar{X})^2$
$(x-\bar{X})^2$	0.0123	4.4567	37.3455	0.7901	15.1235	3.5679	8.3457	0.0123	1.2345	8.8611

$$S = \sqrt{8.8611} \approx 2.9768$$

1. Confidence interval for the mean in the case of unknown mean could be calculated as follows:

$$\bar{X} \pm t_{8,0.05}\frac{S}{\sqrt{10}}$$

The necessary assumption that allows to use t-distribution is that population is normally distributed.

$$t_{8,0.05} = 1.86$$

$$\text{95\% confidence interval is } \; 4.8889 \pm 1.86\frac{2.9768}{\sqrt{3}} \quad \text{or } [3.04, 6.73]$$

2.

$$t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}} = \frac{4.8889-5}{2.9768/3} \approx -0.1120$$

$$t_{8,0.05} = 1.86$$

Since $-1.86 < -0.1120$ we fail to reject H_0

Problem 5 (15 points)

Bob earns on average 70 dollars per day. However, some days he earns more than on other days. It could be assumed that his daily income has a Normal distribution with mean 70 and standard deviation of 15 dollars, i.e. $W_B \sim N(70, 15^2)$. His wife Mary earns on average 80 dollars with standard deviation of 20 : $W_M \sim N(80, 20^2)$. Correlation between W_B and W_M is (-0.40).

Each time when Bob and Mary earn together more than 200 dollars ($W_M + W_B > 200$) they go to their favorite restaurant. Find the probability that at a given day they would go to their favorite restaurant.

$$E(W_B + W_M) = E(W_B) + E(W_M) = 70 + 80 = 150$$

$$Var(W_B + W_M) = Var(W_B) + Var(W_M) + 2 * Corr * \sqrt{Var(W_B)}\sqrt{Var(W_M)} =$$

$$= \sigma_X^2 + \sigma_Y^2 + 2 * \rho * \sigma_X * \sigma_Y = 225 + 400 - 0.40 * 2 * 15 * 20 = 385$$

$$P((W_B + W_M) > 200) = P\left(\frac{(W_B + W_M) - E(W_B + W_M)}{\sqrt{Var(W_B + W_M)}} > \frac{200 - E(W_B + W_M)}{\sqrt{Var(W_B + W_M)}}\right) =$$

$$= P\left(\frac{(W_B + W_M) - 150}{\sqrt{385}} > \frac{200 - 150}{\sqrt{385}}\right) = P\left(Z > 2.5482\right) = 1 - P(Z \leq 2.5482) =$$

$$\approx 1 - F(2.55) \approx 1 - 0.9946 \approx 0.0054$$

Problem 6 (20 points)

Alfred works as a junior consultant in one of the DC consulting firms. One day, his Manager Steve asks him to investigate how significant is impact of monthly income changes on average monthly expenditures on consumer electronics. To answer this question, firms reviews information for 102 randomly chosen people. Alfred receives the following information from a data collection team:

$n = 102$, $S_X = 1200$, $S_Y = 100$, $\bar{X} = 4600$, $\bar{Y} = 400$, $Corr(X, Y) = 0.7$, $S_e^2 = 15000$, $\sum_{i=1}^{n} x_i^2 \approx 5120^2$

a. Alfred never took 390 at Penn State and does not know what to do. Help him to estimate coefficients b_0 and b_1 of the following regression equation: $y_i = b_0 + b_1 * X_i + \varepsilon_i$, where $\varepsilon_i \sim N(0, \sigma^2)$.

b. Find estimates of standard deviation of $\hat{b}_0$ and $\hat{b}_1$.

c. Test $H_0 : b_1 = 0$ against $H_1 : b_1 \neq 0$ at 5% level

d. Do you have information to calculate R^2? If "Yes" calculate it, if "not" state explicitly what additional information you would need.

a.

$$\hat{b}_1 = \frac{Corr(X, Y) * S_Y}{S_X} = \frac{0.7 * 100}{1200} \approx 0.0583$$

$$\hat{b}_0 = \bar{Y} - \bar{X} * \hat{b}_1 = 400 - 4600 * 0.0583 \approx 131.82$$

b.

$$\hat{S}_{\hat{b}_1} = \sqrt{\frac{S_e^2}{\sum_{i=1}^{n}(x_i - \bar{X})^2}} \approx \sqrt{\frac{15000}{101 * 1200^2}} \approx 0.0102$$

$$\hat{S}_{\hat{b}_0} = \sqrt{\frac{S_e^2 \sum_{i=1}^{n} x_i^2}{n \sum_{i=1}^{n}(x_i - \bar{X})^2}} \approx \sqrt{\frac{15000 * 5120^2}{102 * 101 * 1200^2}} \approx \sqrt{26.5062} \approx 5.15$$

c.

$$t = \frac{\hat{b}_1}{\hat{S}_{\hat{b}_1}} = \frac{0.0583}{0.0102} \approx 5.72$$

$$t_{0.025,100} \approx 1.96$$

Since $5.72 > 1.96$ we have to reject H_0

d.

$$R^2 = Corr(X, Y)^2 = 0.7^2 = 0.49$$

Problem 7 (15 points)

A company that receives shipments of fruits from South America takes at random 16 boxes, 100 pounds each. Company has a rule that it accepts the shipment if mean weight of non-fresh food in each box for the whole shipment (μ) is not more than 5 pounds. From the previous experience firm concludes that distribution of fresh food in box is approximately normal with standard deviation is 1 pound.

For one particular shipment firm calculates that for 16 boxes average weight of non-fresh fruits $\bar{X} = 5.4$.

a. Test at 10% level the null hypothesis that the population mean $\mu \leq 5$ against the alternative $\mu > 5$. Should company accept this particular shipment?

b. Redo test from part a at 1% level. Do you have a different conclusion about what the firm have to do?

c. Find the power of the test described in part a for $\mu^* = 5.15$

a.

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{5.4 - 5}{1/\sqrt{16}} = 1.6$$

$$Z_{0.10} = 1.28$$

We can reject H_0 in favor or H_1

b.

$$Z_{0.01} = 2.34$$

We fail to reject H_0 in favor of H_1

c.

$$\beta = P(\bar{X} \leq X_C \mid \mu^* = 5.15) =$$

$$= P\left(\frac{\bar{X} - \mu^*}{\sigma/\sqrt{n}} \leq \frac{X_C - \mu^*}{\sigma/\sqrt{n}} \mid \mu^* = 5.15\right) = F\left(\frac{X_C - \mu^*}{\sigma/\sqrt{n}}\right) =$$

$$= F\left(\frac{\bar{X} + Z_\alpha \frac{\sigma}{\sqrt{n}} - \mu^*}{\sigma/\sqrt{n}}\right) = F\left(\frac{5.0 + 1.28 * \frac{1}{\sqrt{16}} - 5.15}{1/\sqrt{16}}\right) = F\left(\frac{0.17}{0.25}\right) = F(0.68) = 0.7517$$

Power of the test is $1 - \beta = 0.2483$