The Pennsylvania State University

Department of Economics

Econ 390, Section 001, Summer 2007

Homework Assignment # 2

SOLUTION

Problem 1 (Warm up)

Suppose that

$$E(X) = \mu_X = 0.75$$

$$Var(X) = \sigma_X^2 = 0.30$$

$$E(Y) = \mu_Y = 1.05$$

$$Var(Y) = \sigma_Y^2 = 0.25$$

a) $E(X+Y) = 0.75 + 1.05 = 1.80$

b) $E(X-Y) = 0.75 - 1.05 = -0.3$

c) $E(2X) = 2 * 0.75 = 1.50$

d) $E(-3Y) = -3 * 1.05 - 3.15$

e) $Var(3+2X) = 4 * Var(X) = 4 * 0.30 = 1.20$

f) $Var(1-4Y) = 16 * Var(Y) = 4$

g) S has expectation 0 and Variance 1, since:

$$E(S) = E\left(\frac{X - 0.75}{\sqrt{0.30}}\right) = \frac{1}{\sqrt{0.30}} E\left(X - 0.75\right) = \frac{1}{\sqrt{0.30}} \left(E\left(X\right) - (0.75)\right) = 0$$

$$Var(S) = Var\left(\frac{X - 0.75}{\sqrt{0.30}}\right) = Var\left(\frac{X}{\sqrt{0.30}} - \frac{0.75}{\sqrt{0.30}}\right) = Var\left(\frac{X}{\sqrt{0.30}}\right) = \frac{0.30}{0.30} = 1$$

Problem 2.

Suppose that Bill is an owner of a small restaurant in NY After reviewing his business accounting data he finds out that his daily revenues can be summarized by the following table:

Revenue	*PDF*	*CDF*	$x*p(x)$	$(x-\mu_X)^2\,p(x)$	$x^2p(x)$	$x^2p(x)-(\mu_X^2/7)$
1200	0.15	0.15	180	$538,653.75$	216000	$216000 - 3095\hat{}^{2} * (1/7)$
2500	0.20	0.35	500	$70,81$	1250000	$1250000 - 3095\hat{}^{2} * (1/7)$
3100	0.15	0.50	465	3.75	1441500	$1441500 - 3095\hat{}^{2} * (1/7)$
3500	0.30	0.80	1050	$49,207.50$	3675000	$3675000 - 3095\hat{}^{2} * (1/7)$
4000	0.10	0.90	400	$81,902.50$	1600000	$1600000 - 3095\hat{}^{2} * (1/7)$
4500	0.05	0.95	225	$98,701.25$	1012500	$1012500 - 3095\hat{}^{2} * (1/7)$
5500	0.05	1.00	275	$289,201.25$	1512500	$1512500 - 3095\hat{}^{2} * (1/7)$
SUM			3095	$1,128,475.00$	10707500	$10707500 - 3095\hat{}2$

1) Plot *PDF*

See the last page

2) Plot *CDF* (Hint: First fill out free cells in the table - it will help you answer questions 2 – 5)

See the last page

3) Calculate mean revenue. Recall that $\mu_X = \sum_{x \in X} x * p(x)$

From the column 4, it is 3095

4) Calculate variance and standard deviation using expression:

$$\sigma_X^2 = \sum_{x \in X} (x - \mu_X)^2\, p(x)$$

From the column 5, Variance is $1,128,475.00$

Standard deviation is $\sqrt{1,128,475.00} \approx 1062$

5) Calculate variance using expression

$$\sigma_X^2 = \left[\sum_{x \in X} x^2 p(x)\right] - \mu_X^2$$

From columns 6 and 7:

$$\sigma_X^2 = \left[\sum_{x \in X} x^2 p(x)\right] - \mu_X^2 = 10707500 - 3095\hat{}2 = 1,128,475.00$$

Standard deviation is $\sqrt{1,128,475.00} \approx 1062$

6) Suppose that Bill wants to send a copy of the report to his friend in Great Britain. To help his friend he converts all numbers into GB pounds. Exchange rate is 1 dollar = 0.5 pounds.

a) Find the mean revenue in GB pounds

E(Revenue in pounds)= 0.5*E(Revenue in dollars)= $0.5 * 3095$=1547.5

b) Calculate the standard deviation of the revenue in GB pounds

Var(Revenue in pounds)= 0.5^2*Var(Revenue in dollars)= $0.25 * 1,128,475.00 = 282118.75$

Standard deviation= $\sqrt{282118.75} \approx 531$

Problem 3

From his previous experience, manager of a car dealership knows that probability that a client will buy a car is 40%. Suppose one day 10 people who do not know each other enter the dealership.

Find the following probabilities:

1) $P(exactly\ 5\ cars\ sold)$

We use Binomial distribution to solve for the probability:

$$P(5) = C_5^{10}(0.40)^5(1-0.40)^{10-5} = \frac{10!}{5!5!}(0.40)^5(0.60)^5 \approx 0.20$$

2) $P(8\ or\ more\ cars\ sold)$

$$P(X \geqslant 8) = P(X = 8) + P(X = 9) + P(X = 10) =$$

$$= C_8^{10}(0.40)^8(1-0.40)^{10-8} + C_9^{10}(0.40)^9(1-0.40)^{10-9} + C_9^{10}(0.40)^9(1-0.40)^{10-9} =$$

$$= 0.0106 + 0.0016 + 0.0001 = 0.0123$$

3) $P(2\ or\ less\ cars\ sold)$

$$P(X \leqslant 2) = P(X = 0) + P(X = 1) + P(X = 2) =$$

$$= C_0^{10}(0.40)^0(1-0.40)^{10-0} + C_1^{10}(0.40)^1(1-0.40)^{10-1} + C_2^{10}(0.40)^2(1-0.40)^{10-2} =$$

$$= 0.0060 + 0.0403 + 0.1209 = 0.1673$$

4) No cars are sold

$$P(X = 0) = C_0^{10}(0.40)^0(1-0.40)^{10-0} = 0.0060$$

Problem 4

A car insurance company has collected data on the number of car accidents per day for the last two years. For the last 2 years on average 5 accidents occurred per day.

a) What is the probability that exactly 7 accidents will happen in a given day?

This problem is based on the Poisson distribution:

$$P(X) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$\lambda = 5$

$$P(5) = \frac{e^{-5}5^7}{7!} \approx 0.1044$$

b) What is the probability that more that 3 accidents will happen in a given day?

Lets define a complement to the event "more that 3 accidents will happen in a given day":
"Less or 3 accidents happen in a given day"

P("Less or exactly 3 accidents happen in a given day")=

$$P("Less...") = P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) =$$
$$= \frac{e^{-5}5^0}{0!} + \frac{e^{-5}5^1}{1!} + \frac{e^{-5}5^2}{2!} + \frac{e^{-5}5^3}{3!} =$$

$$= 0.0067 + 0.0337 + 0.0842 + 0.1404 = 0.2650$$

$$P(X > 3) = 1 - P(X \leq 3) = 1 - 0.2650 = 0.7350$$

c) What is the probability that no accidents will happen in a give day?

$$P(X = 0) = \frac{e^{-5}5^0}{0!} = 0.0067$$

d) Suppose that for each accident company pays on average 3200 dollars. How much does the company pay on average each day? Find standard deviation of this payment.

$$E(X) = \lambda = 5$$

$$E(Payment) = E(3200 * X) = 3200 * E(X) = 3200 * 5 = 16000$$

$$Var(X) = \sigma_X^2 = \lambda = 5$$

$$Var(Payment) = Var(3200 * X) = 3200^2 * Var(X)$$

or

$$\sigma_{payment} = 3200 * \sigma_X = 3200 * 5 = 16000$$

Problem 5

A small commuter airline flies plans that can seat up to eight passengers. The airline has determined that the probability that a ticketed passenger will not show up for a flight is 0.2. For each flight the airline sells tickets to the first 10 people placing orders. The probability distribution for the number of tickets per flight is shown in the accompanying table. For what proportion of the airline's flights does the number of ticketed passengers showing up exceed the number of available seats. (Assume independence between the number of tickets sold and the probability that a ticketed passenger will show up).

Number of tickets	6	7	8	9	10
Probability	0.25	0.35	0.25	0.10	0.05

Note that if company sells 8 or less tickets than it can not be overbooked. The number of passangers showing up may exceed the number of seats only if 9 or 10 tickets are sold.

Suppose 9 tickets are sold. The flight will be overbooked if all 9 people show up. What is probability that exactly 9 people will show up? We use binomial distrivution:

Probability that somebody who buys a tickets shows up is $1 - 0.2 = 0.8$

$$P(9) = C_9^9(0.80)^9(0.20)^0 = 0.1342$$

Suppose 10 tickets are sold. The flight will be overbooked if all 10 or at least 9 people show up:

$$P(9) = C_9^{10}(0.80)^9(0.20)^{10-9} = 0.2684$$

$$P(10) = C_{10}^{10}(0.80)^{10}(0.20)^{10-10} = 0.1074$$

Since we assume independence between the number of tickets sold and the probability that a ticketed passenger will show up the probability that flight is overbooked is:

$$P(Overbooked) = P(9\ tickets\ sold) * P(9\ people\ show\ up) +$$

$$+P(10 tickets sold) * [P(10\ people\ show\ up) + P(9\ people\ show\ up)]$$

From the table $P(9\ tickets\ sold) = 0.10$ and $P(10\ tickets\ sold) = 0.05$, hence:

$$P(Overbooked) = 0.10 * 0.1342 + 0.05 * (0.2684 + 0.1074) = 0.0322$$

Problem 6

The IRS reported that 5.5% of all taxpayers filling out the 1040 short form make mistakes. If 100 of these forms are chosen at random, what is the probability that fewer that 3 of them contain errors? (Hint: Use the Poisson approximation to the binomial distribution).

To solve the problem the best way is to use the Poisson distribution to approximate the Binomial one.
Note that we can do this since condition $np \leq 7$ is satisfied:

$$np = 0.055 * 100 = 5.5 < 7$$

We have to take $\lambda = np = 5.5$

$$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) =$$

$$= \frac{e^{-5.5} 5.5^0}{0!} + \frac{e^{-5.5} 5.5^1}{1!} + \frac{e^{-5.5} 5.5^2}{2!} =$$

$$= 0.0041 + 0.0225 + 0.0618 = 0.0884$$

Problem 7 (Uniform distribution).

Suppose you are driving a car from State College to Washington, DC. The distance between these two cities is 200 miles.
Assume that the probability of your car breaking down is uniform with a density function $f(x) = 0.005$
What is the probability that your car breaks down between 100 to 150 miles away from State College?

Since distribution is uniform probability that car brakes down between 100 to 150 miles away from State College is just

$$\frac{150 - 100}{200} = \frac{50}{200} = 0.25$$

because,

$$\int_{100}^{150} 0.005 dx = 0.005 * x|_{100}^{150} = 0.005 * (150 - 100) = 0.25$$

Problem 8 (Normal distribution)

Let the random variable Z follow a standard normal distribution

a) Find $P(Z < 1.20) = 0.5 + 0.3849 = 0.8849$

b) Find $P(Z > 1.33) = 1 - P(Z < 1.33) = 1 - 0.5 - 0.4082 = 0.0918$

c) Find $P(Z < -1.70) = 0.5 - 0.4554 = 0.0446$

d) Find $P(Z > -1.00) = P(Z \leq 1) = 0.5 + 0.3413 = 0.8413$

e) Find $P(1.20 < Z < 1.33) = 0.5 + 0.4082 - 0.5 - 0.3849 = 0.0233$